

# WEAKLY COUPLED BOUND STATE OF 2D SCHRÖDINGER OPERATOR WITH POTENTIAL-MEASURE

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ABSTRACT. We consider a self-adjoint two-dimensional Schrödinger operator  $H_{\alpha\mu}$ , which corresponds to the formal differential expression

$$-\Delta - \alpha\mu,$$

where  $\mu$  is a finite compactly supported positive Radon measure on  $\mathbb{R}^2$  from the generalized Kato class and  $\alpha > 0$  is the coupling constant. It was proven earlier that  $\sigma_{\text{ess}}(H_{\alpha\mu}) = [0, +\infty)$ . We show that for sufficiently small  $\alpha$  the condition  $\sharp\sigma_{\text{d}}(H_{\alpha\mu}) = 1$  holds and that the corresponding unique eigenvalue has the asymptotic expansion

$$\lambda(\alpha) = -(C_\mu + o(1)) \exp\left(-\frac{4\pi}{\alpha\mu(\mathbb{R}^2)}\right), \quad \alpha \rightarrow 0+,$$

with a certain constant  $C_\mu > 0$ . We obtain also the formula for the computation of  $C_\mu$ . The asymptotic expansion of the corresponding eigenfunction is provided. The statements of this paper extend Simon's results, see [Si76], to the case of potentials-measures. Also for regular potentials our results are partially new.

## 1. INTRODUCTION

Let us consider a non-relativistic quantum particle living in a two-dimensional system and moving under the influence of the potential  $V: \mathbb{R}^2 \rightarrow \mathbb{R}$  such that there exists  $\delta > 0$  for which

$$(1.1) \quad \int_{\mathbb{R}^2} |V(x)|^{1+\delta} < \infty \quad \text{and} \quad \int_{\mathbb{R}^2} |V(x)|(1 + |x|^\delta) < \infty.$$

The operator

$$H_{\alpha V} = -\Delta - \alpha V : \text{dom } H_{\alpha V} \rightarrow L^2(\mathbb{R}^2)$$

is self-adjoint with  $\text{dom } H_{\alpha V} = H^2(\mathbb{R}^2)$  and it determines the Hamiltonian of our system. This operator represents the sesquilinear form

$$\mathfrak{t}_{\alpha V}[f, g] := (\nabla f, \nabla g)_{L^2(\mathbb{R}^2; \mathbb{C}^2)} - \alpha(Vf, g)_{L^2(\mathbb{R}^2)}, \quad \text{dom } \mathfrak{t}_{\alpha V} := H^1(\mathbb{R}^2).$$

The spectrum of  $H_{\alpha V}$  can not be computed explicitly for an arbitrary potential. For this reason spectral estimates and asymptotic expansions of spectral quantities related to  $H_{\alpha V}$  attract a lot of attention. Weak coupling

asymptotic regime belongs to this line of research. It was shown by Simon in [Si76] that under the assumptions

$$(1.2) \quad \int_{\mathbb{R}^2} V(x) \geq 0 \quad \text{and} \quad V \neq 0$$

the operator  $H_{\alpha V}$  has at least one bound state for any  $\alpha > 0$ ; moreover for  $\alpha$  small the corresponding lowest eigenvalue asymptotically behaves as

$$\lambda(\alpha) \sim -\exp\left(\left[\frac{\alpha}{4\pi} \int_{\mathbb{R}^2} V(x)\right]^{-1}\right), \quad \alpha \rightarrow 0+,$$

provided inequality (1.2) is sharp; cf. [Si76, Theorem 3.4].

The problem we study in this paper, is addressed in a certain respect to a more general class of potentials which, for example, includes so-called *singular interactions*. To sketch the physical context suppose that a particle is confined by a quantum wire with possibility of tunnelling. Consequently, the whole space  $\mathbb{R}^2$  is available for the particle. On the other hand, if the wire is very thin we can make an idealization and assume that the particle is localized in the vicinity of the set  $\Sigma \subset \mathbb{R}^2$  of a lower dimension. The Hamiltonian of such a system can be formally written as

$$-\Delta - \alpha \delta_{\Sigma}, \quad \alpha > 0,$$

where  $\delta_{\Sigma}$  denotes the Dirac measure supported on  $\Sigma$ , see [E08] for the review on such Hamiltonians. More generally, one can speak of

$$-\Delta - \alpha \mu, \quad \alpha > 0,$$

where  $\mu$  is a positive finite Radon measure on  $\mathbb{R}^2$ . In order to give a mathematical meaning to the above formal expression we assume that  $\mu$  belongs to the generalized Kato class as in Definition 2.1. Under this assumption the embedding of  $H^1(\mathbb{R}^2)$  into  $L^2(\mathbb{R}^2; d\mu)$  is well defined and the following closed, densely defined, symmetric and lower-semibounded sesquilinear form

$$\mathfrak{t}_{\alpha\mu}[f, g] := (\nabla f, \nabla g)_{L^2(\mathbb{R}^2; \mathbb{C}^2)} - \alpha \int_{\mathbb{R}^2} f(x) \overline{g(x)} d\mu(x), \quad \text{dom } \mathfrak{t}_{\alpha\mu} := H^1(\mathbb{R}^2),$$

induces the uniquely defined self-adjoint operator  $H_{\alpha\mu}$  in  $L^2(\mathbb{R}^2)$ . It is known that  $\sigma_{\text{ess}}(H_{\alpha\mu}) = [0, +\infty)$ , see [BEKS94, Theorem 3.1]. The following theorem contains all the main results of the paper.

**Theorem.** *Let  $\mu$  be a compactly supported positive finite Radon measure on  $\mathbb{R}^2$  from the generalized Kato class and  $H_{\alpha\mu}$  be the self-adjoint operator defined above. Then the following statements hold.*

- (i) *For  $\alpha > 0$  sufficiently small we have*

$$\sharp\sigma_{\text{d}}(H_{\alpha\mu}) = 1.$$

*Denote this unique eigenvalue by  $\lambda(\alpha) < 0$  and the corresponding eigenfunction by  $f_{\alpha} \in L^2(\mathbb{R}^2)$ .*

(ii) *The asymptotic expansion of  $\lambda(\alpha)$  takes the form*

$$\lambda(\alpha) = -(C_\mu + o(1)) \exp\left(-\frac{4\pi}{\alpha\mu(\mathbb{R}^2)}\right), \quad \alpha \rightarrow 0+,$$

*where the constant  $C_\mu$  is given in (3.3).*

(iii) *Set  $k_\alpha = (-\lambda(\alpha))^{1/2}$ . Then the corresponding eigenfunction admits the following expansion*

$$f_\alpha(\cdot) = \frac{k_\alpha}{2\pi} \int_{\mathbb{R}^2} K_0(k_\alpha |\cdot - y|) d\mu(y) + O\left(\frac{1}{\ln k_\alpha}\right), \quad \alpha \rightarrow 0+,$$

*where  $K_0(\cdot)$  is the Macdonald function, the norm of the first summand has non-zero finite limit as  $\alpha \rightarrow 0+$ , and the error term is understood in the strong sense.*

The reader may note that in the asymptotic expansion of  $\lambda(\alpha)$  the dominating term depends only on the total measure of  $\mathbb{R}^2$  and does not depend on the distribution character of the measure  $\mu$ . This stays in consistency with Simon's result and reflects the property that in the weak coupling regime spectral quantities “forget” about local properties of the potential.

The statements of this paper constitute the extensions and generalizations of the results obtained in [Si76]. Firstly, the class of perturbations that we admit contains, for example, singular measures as  $\delta$ -distributions supported on sets of lower dimensions. Secondly, for regular compactly supported potentials our class is slightly larger than that of [Si76]. In order to give the reader an idea of that, let us only mention that radially symmetric potential

$$V(r) = \frac{\chi(r)}{r^2 |\ln(r)|^\gamma},$$

with  $\chi(r)$  being the characteristic function of the interval  $[0, 1/2]$  and  $\gamma > 2$ , is compactly supported and belongs to the generalized Kato class, however it does not satisfy assumptions (1.1), which are imposed in [Si76]. One should say that the formula for the constant  $C_\mu$  given in (3.3) is derived formally by physicists [Pa80] in the case of regular potentials, but without a rigorous mathematical proof.

Analogous asymptotic expansions of the bound state with respect to a small parameter appear in various spectral problems. It is worth to mention such results for two-dimensional waveguides with weak local perturbations [BGRS97] as well as for coupled waveguides with a small window [P99] and also with a semi-transparent window [EKr01]. Recently a “leaky waveguide” with a small parameter breaking the symmetry was considered in [KK13]. For the similar problems in the one-dimensional case see [BGS77, K177, LL58, Si76]. The analogous results for quantum graphs were obtained in [EEK10, E96, K07]. See also recent developments for Pauli operators [FMV11]. Our list of references is far from being complete, however many of significant related works are mentioned.

In order to prove the main statements we will apply the Birman-Schwinger principle. Precisely saying, we use its generalization for potentials-measures from the generalized Kato class, which is rigorously established in [BEKS94], see also [Br95] and [P01, BLL13] for further modifications. We also use some simple results of perturbation theory of linear operators, where the standard reference is [K], however we require some extensions of the classical results.

The paper is organized as follows. In Section 2 we complete some mathematical tools useful for further spectral analysis. Namely, we provide a rigorous definition of the Hamiltonian  $H_{\alpha\mu}$ , formulate the Birman-Schwinger principle, develop a perturbation method for a particular class of non-analytic operator families and analyze the properties of the operators involved into the Birman-Schwinger principle. In Section 3 we formulate and prove main results of the paper concerning the uniqueness of the bound state in the weak coupling regime, obtain its asymptotic behavior and derive the behavior of the corresponding eigenfunction.

In the remaining part of the paper we employ the following abbreviations:

- we set  $L^2 := L^2(\mathbb{R}^2)$  (norm  $\|\cdot\|$ ),  $L^1 := L^1(\mathbb{R}^2)$ ,  $H^k \equiv H^k(\mathbb{R}^2)$  with  $k \in \mathbb{Z}$  (norm  $\|\cdot\|_k$ ) and  $\mathbb{L}^2 := L^2(\mathbb{R}^2; \mathbb{C}^2)$ ;
- the notation  $\mathcal{S} := \mathcal{S}(\mathbb{R}^2)$  stands for the Schwartz class, moreover we set  $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^2)$  for the space dual to  $\mathcal{S}$ , i.e.  $\mathcal{S}'$  is the space of linear continuous functionals on  $\mathcal{S}$ ;
- we set  $L_\mu^2 := L^2(\mathbb{R}^2; d\mu)$  and  $L_\mu^1 := L^1(\mathbb{R}^2; d\mu)$ ;
- for the positive Radon measure  $\mu$  on  $\mathbb{R}^2$  we denote  $\mu_T := \mu(\mathbb{R}^2)$ .

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## 2. PRELIMINARIES

This section plays an auxiliary role and consists of four subsections. In Subsection 2.1 we provide necessary facts from [BEKS94, Br95] on self-adjoint free Laplacians perturbed by Kato-class measures. In Subsection 2.2 we prove some statements on non-analytic perturbation theory, which are hard to find in the literature. In Subsections 2.3 and 2.4 we complement known results on the operators related to Birman-Schwinger principle.

**2.1. Self-adjoint Laplacians perturbed by Kato-class measures.** We start with recalling the definition of the generalized Kato class of positive Radon measures on  $\mathbb{R}^2$ .

**Definition 2.1.** *A positive Radon measure  $\mu$  on  $\mathbb{R}^2$  belongs to the generalized Kato class if*

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{x \in \mathbb{R}^2} \int_{D_\varepsilon(x)} |\ln|x-y|| d\mu(y) = 0,$$

where  $D_\varepsilon(x)$  is the disc of radius  $\varepsilon > 0$  with the center at  $x \in \mathbb{R}^2$ .

Let  $\mu$  be a positive Radon measure from the generalized Kato class. Then for arbitrarily small  $\varepsilon > 0$  there exists a constant  $C(\varepsilon) > 0$  such that

$$\int_{\mathbb{R}^2} |f(x)|^2 d\mu(x) \leq \varepsilon \|\nabla f\|_{\mathbb{L}^2}^2 + C(\varepsilon) \|f\|_{L^2}^2$$

holds for every  $f \in \mathcal{S}$ ; see [BEKS94, SV96]. For the measure  $\mu$  the embedding operator  $J_\mu: H^1 \rightarrow L_\mu^2$  is well-defined as the closure of the natural embedding defined on the Schwartz class, see [BEKS94, Section 2]. Consequently, the above inequality has a natural extension, i.e. for arbitrarily small  $\varepsilon > 0$  there exists a constant  $C(\varepsilon) > 0$  such that

$$(2.1) \quad \|J_\mu f\|_{L_\mu^2}^2 \leq \varepsilon \|\nabla f\|_{\mathbb{L}^2}^2 + C(\varepsilon) \|f\|_{L^2}^2,$$

for all  $f \in H^1$ .

**Example 2.1.** Suppose that the measurable function  $V: \mathbb{R}^2 \rightarrow [0, +\infty)$  satisfies the condition

$$\lim_{\varepsilon \rightarrow 0+} \sup_{x \in \mathbb{R}^2} \int_{D_\varepsilon(x)} |\ln|x-y|| V(y) dy = 0.$$

Then the measure

$$\mu_V(\Omega) := \int_{\Omega} V(x) dx$$

belongs to the generalized Kato class.

**Example 2.2.** [V09, Example 2.3 (c)] Given a family  $\{\Gamma_i\}_{i=1}^N$  of Lipschitz curves in the plane. Suppose that each curve in the family is parameterized by its arc length  $\gamma_i: [0, |\Gamma_i|] \rightarrow \mathbb{R}^2$  and  $\gamma_i([0, |\Gamma_i|]) = \Gamma_i$  with  $i = 1, 2, \dots, N$ . Assume that there exist  $c \in (0, 1]$  such that for all  $s, t \in [0, |\Gamma_i|]$  the condition  $|\gamma_i(s) - \gamma_i(t)| \geq 1/2|s - t|$  holds with  $i = 1, 2, \dots, N$ . So that each curve can not have cusps and can not intersect itself, whereas different curves can intersect each other. Now let  $\Gamma := \cup_{i=1}^N \Gamma_i$ . Then the Dirac measure supported on  $\Gamma$  belongs to the generalized Kato class.

Let the self-adjoint operator

$$-\Delta : \text{dom}(-\Delta) \rightarrow L^2, \quad \text{dom}(-\Delta) = H^2,$$

define the unperturbed Hamiltonian of our system. In fact,  $-\Delta$  represents closed, densely defined, symmetric and lower-semibounded sesquilinear form

$$(2.2) \quad \mathfrak{t}[f, g] = (\nabla f, \nabla g)_{\mathbb{L}^2}, \quad \text{dom } \mathfrak{t} = H^1.$$

Let  $\mu$  be a positive Radon measure from the generalized Kato class. By means of  $\mu$  we define the sesquilinear form

$$(2.3) \quad \mathfrak{t}_{\alpha\mu}[f, g] := \mathfrak{t}[f, g] - (\alpha J_\mu f, J_\mu g)_{L_\mu^2}, \quad \text{dom } \mathfrak{t}_{\alpha\mu} := H^1,$$

which, in view of (2.1) and KLMN-theorem, cf. [RS-II, Theorem X.17], is symmetric, closed and lower-semibounded.

**Definition 2.2.** Let  $H_{\alpha\mu}$  be a self-adjoint operator acting in  $L^2$  and defined as the operator associated with  $\mathfrak{t}_{\alpha\mu}$  via the first representation theorem, [K, Chapter VI, Theorem 2.1].

Denote  $R(\lambda) := (-\Delta - \lambda)^{-1}$  with  $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ . Then  $R(\lambda)$  is an integral operator with the kernel

$$G(x, y; \lambda) = \frac{1}{2\pi} K_0(i\sqrt{\lambda}|x - y|), \quad x, y \in \mathbb{R}^2,$$

where  $K_0(\cdot)$  is the Macdonald function, see [AS64, §9.6]. Following the notations of [BEKS94] we introduce the integral operator

$$(2.4) \quad R_{\mu dx}(\lambda): L_\mu^2 \rightarrow L^2, \quad R_{\mu dx}f := \int_{\mathbb{R}^2} G(x, y; \lambda) f(y) d\mu(y),$$

and define the “bilateral” embedding of  $R(\lambda)$  to  $L_\mu^2$  by

$$(2.5) \quad Q(\lambda) := J_\mu R_{\mu dx}(\lambda): L_\mu^2 \rightarrow L_\mu^2.$$

Note that

$$(2.6) \quad Q(-k^2)f := \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(k|\cdot - y|) f(y) d\mu(y).$$

The Birman-Schwinger principle takes the following form.

**Proposition 2.3.** [Br95, Lemma 1], [BEKS94] Let  $R_{\mu dx}(\cdot)$ ,  $Q(\cdot)$  and  $H_{\alpha\mu}$  be as above. For  $\lambda \in \mathbb{R}_-$  the mapping

$$h \mapsto R_{\mu dx}(\lambda)h$$

is a bijection from  $\ker(I - \alpha Q(\lambda))$  onto  $\ker(H_{\alpha\mu} - \lambda)$ , and

$$\dim \ker(I - \alpha Q(\lambda)) = \dim \ker(H_{\alpha\mu} - \lambda).$$

We will also use the fact that the essential spectrum is stable under a perturbation of a finite measure.

**Proposition 2.4.** [BEKS94, Theorem 3.1] Let  $\mu$  be a positive Radon measure on  $\mathbb{R}^2$  from the generalized Kato class. Assume that  $\mu_T < \infty$  and  $H_{\alpha\mu}$  is as in Definition 2.2. Then

$$\sigma_{\text{ess}}(H_{\alpha\mu}) = [0, +\infty)$$

holds.

*Remark 2.5.* Note that also more singular perturbations are considered. For example,  $\delta$ -interactions supported on curves in  $\mathbb{R}^3$ , see [EK02, EK03, EK08, K12, P01], and  $\delta'$ -interactions supported on hypersurfaces, see [BEL13, BLL13, EJ13]. These perturbations do not belong to the generalized Kato class and therefore they require different approaches.

**2.2. Elements of non-analytic perturbation theory.** Putting in mind later purposes we analyse a family of self-adjoint operators  $k \mapsto T(k)$ ,  $k \in \mathbb{R}_+$ , acting in a Hilbert space  $\mathcal{H}$  and taking the form

$$T(k) := T_0 + \frac{1}{\ln k} T_1 + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+,$$

where  $T_0 = \varphi(\cdot, \varphi)$  with  $\varphi \in \mathcal{H}$  being a normalized function,  $T_1$  is a bounded self-adjoint operator in  $\mathcal{H}$  and the error term is understood in the operator norm sense. The family  $T(\cdot)$  is not analytic and consequently we can not apply directly the results of [K, Chapters II and VII]. In the following theorem we investigate the spectra and the eigenfunctions of  $T(k)$  in the limit  $k \rightarrow 0+$ .

**Theorem 2.6.** *Let  $k \mapsto T(k)$  be defined as above. For sufficiently small  $k > 0$  the spectrum  $\sigma(T(k)) \subset \mathbb{R}$  of  $T(k)$  consists of two disjoint components  $\sigma_0(k)$  and  $\sigma_1(k)$ .*

- (i) *The part  $\sigma_0(k)$  is located in the small neighborhood of zero and its diameter can be estimated as*

$$\text{diam } \sigma_0(k) \leq \frac{1}{|\ln k|} \|T_1\| + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

- (ii) *The part  $\sigma_1(k)$  consists of exactly one eigenvalue  $\omega(k)$  of multiplicity one, which depends on  $k$  continuously.*  
 (iii) *The normalized eigenfunction  $\varphi_k$  corresponding to the eigenvalue  $\omega(k)$  has the following expansion*

$$(2.7) \quad \varphi_k = \varphi + O\left(\frac{1}{\ln k}\right), \quad k \rightarrow 0+,$$

*in the norm of  $\mathcal{H}$ .*

- (iv) *The eigenvalue  $\omega(k)$  admits the asymptotics*

$$(2.8) \quad \omega(k) = 1 + \frac{1}{\ln k} (T_1 \varphi, \varphi) + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

*Proof.* (i) Note that  $\sigma(T_0) = \{0, 1\}$  and that  $\varphi$  is an eigenfunction of the operator  $T_0$  corresponding to the eigenvalue 1. The separation of the spectra of  $T(k)$  into two parts  $\sigma_0(k)$  and  $\sigma_1(k)$  for sufficiently small  $k > 0$  follows from [K, Theorem V.4.10]. The component  $\sigma_0(k)$  is located in the neighborhood of 0 and the component  $\sigma_1(k)$  is located in the neighborhood of 1. Note that again by [K, Theorem V.4.10] the diameter of  $\sigma_0(k)$  satisfies

$$\text{diam } \sigma_0(k) \leq \frac{1}{|\ln k|} \|T_1\| + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

(ii) Let  $E_i(k)$ ,  $i = 0, 1$ , be the orthogonal projectors onto the spectral subspaces of the operator  $T(k)$  corresponding to  $\sigma_i(k)$ . Then  $E_0(0) = I - T_0$  and  $E_1(0) = T_0$  hold. Since  $\|T(k) - T_0\|$  tends to 0 for  $k \rightarrow 0+$ , relying on [DD87, Theorem 3] we have  $\dim \text{ran } E_1(k) = 1$  for sufficiently small  $k > 0$ . Therefore  $E_1(k) = \tilde{\varphi}_k(\cdot, \tilde{\varphi}_k)$ , where  $\tilde{\varphi}_k$  is the normalized eigenfunction corresponding to the eigenvalue  $\omega(k)$  of  $T(k)$  with multiplicity one. According to [K, Theorem VIII.1.14] the eigenvalue  $\omega(k)$  depends on  $k$  continuously.

(iii) By [KMM07, Proposition 2.1], see also [BDM83], the estimate

$$\text{dist}(\sigma_0(k), \sigma_1(k)) \|E_0(k)E_1(0)\| \leq \frac{\pi}{2} \|T(k) - T(0)\|$$

holds, which yields the asymptotic property

$$\|E_1(0) - E_1(k)E_1(0)\| = O\left(\frac{1}{\ln k}\right), \quad k \rightarrow 0+,$$

where we have used  $E_0(k) = I - E_1(k)$ . The above expansion implies the following

$$(2.9) \quad \|\varphi - \tilde{\varphi}_k(\varphi, \tilde{\varphi}_k)\| = O\left(\frac{1}{\ln k}\right), \quad k \rightarrow 0+.$$

A straightforward calculation yields

$$\begin{aligned} \|\varphi - \tilde{\varphi}_k(\varphi, \tilde{\varphi}_k)\|^2 &= (\varphi - \tilde{\varphi}_k(\varphi, \tilde{\varphi}_k), \varphi - \tilde{\varphi}_k(\varphi, \tilde{\varphi}_k)) \\ &= 1 - (\tilde{\varphi}_k, \varphi)(\varphi, \tilde{\varphi}_k) - (\varphi, \tilde{\varphi}_k)\overline{(\varphi, \tilde{\varphi}_k)} + |(\varphi, \tilde{\varphi}_k)|^2 \\ &= 1 - |(\tilde{\varphi}_k, \varphi)|^2. \end{aligned}$$

Combining the above result with the estimate (2.9) we arrive at

$$(2.10) \quad 1 - |(\tilde{\varphi}_k, \varphi)|^2 = O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

Consequently, we obtain

$$(2.11) \quad 1 - |(\tilde{\varphi}_k, \varphi)| = O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

Suppose that  $(r(k), \theta(k))$  determine the polar representation of  $(\tilde{\varphi}_k, \varphi)$ , i.e.  $(\tilde{\varphi}_k, \varphi) = r(k)e^{i\theta(k)}$ . According to (2.11) we claim that

$$(2.12) \quad r(k) = 1 + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

Since  $\tilde{\varphi}_k$  is the normalized eigenfunction of  $T(k)$  corresponding to the eigenvalue  $\omega(k)$  the function

$$(2.13) \quad \varphi_k := e^{i\theta(k)}\tilde{\varphi}_k$$

is as well. Thence, by (2.9) and (2.12) we get

$$\begin{aligned} \|\varphi - \varphi_k\| &= \|\varphi - e^{i\theta(k)}\tilde{\varphi}_k\| \leq \|\varphi - r(k)e^{i\theta(k)}\tilde{\varphi}_k\| + \|r(k)e^{i\theta(k)}\tilde{\varphi}_k - e^{i\theta(k)}\tilde{\varphi}_k\| \\ &= \|\varphi - (\varphi, \tilde{\varphi}_k)\tilde{\varphi}_k\| + |r(k) - 1| = O\left(\frac{1}{\ln k}\right), \quad k \rightarrow 0+, \end{aligned}$$

which proves the expansion (2.7).

(iv) Moreover,  $\omega(k) \in \sigma_1(k)$  as an eigenvalue of  $T(k)$  with multiplicity one admits the representation

$$\omega(k) = \left(T(k)\varphi_k, \varphi_k\right) = \left(T_0\varphi_k, \varphi_k\right) + \frac{1}{\ln k} \left(T_1\varphi_k, \varphi_k\right) + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

Applying (2.7) and the fact that  $T_1$  is bounded, we get

$$\omega(k) = |(\varphi, \varphi_k)|^2 + \frac{1}{\ln k} (T_1\varphi, \varphi) + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

Using (2.10) and (2.13) we get the asymptotics of  $\omega(\cdot)$  given in (2.8).  $\square$



**2.3. Properties of the  $Q(\cdot)$ -function.** In this subsection we analyze the operator-valued function  $Q(\cdot)$  defined in (2.5). Our aim is to describe certain basic properties of  $Q(\cdot)$  and to derive its asymptotic expansion in the neighborhood of zero. The following lemma provides the first auxiliary tool.

**Lemma 2.7.** *Let  $\mu$  be a compactly supported positive finite Radon measure on  $\mathbb{R}^2$  belonging to the generalized Kato class and  $C \in \mathbb{R}$  be a constant. Then the integral operator acting as*

$$Rf := \int_{\mathbb{R}^2} (-\ln |\cdot - y| + C) f(y) d\mu(y)$$

*is bounded in  $L_\mu^2$ .*

*Proof.* The operator  $R$  can be decomposed into the sum of two integral operators:

$$R_1 f = \int_{\mathbb{R}^2} (-\ln |\cdot - y|) f(y) d\mu(y), \quad R_2 f := C \int_{\mathbb{R}^2} f(y) d\mu(y).$$

According to the definition of the generalized Kato class (Definition 2.1) for any constant  $A > 0$  one can find  $\varepsilon > 0$  such that for every  $x_0 \in \text{supp } \mu$  the estimate

$$(2.14) \quad \int_{D_\varepsilon(x_0)} |\ln |x_0 - y|| d\mu(y) \leq A$$

holds. Hence for any  $x_0 \in \text{supp } \mu$  we get

$$\begin{aligned} & \int_{\mathbb{R}^2} |\ln |x_0 - y|| d\mu(y) \\ &= \int_{D_\varepsilon(x_0)} |\ln |x_0 - y|| d\mu(y) + \int_{\mathbb{R}^2 \setminus D_\varepsilon(x_0)} |\ln |x_0 - y|| d\mu(y) \\ &\leq A + \max \{ |\ln |\varepsilon||, |\ln |\text{diam supp } \mu|| \} \mu_T. \end{aligned}$$

Note that the bound above is independent of the choice of  $x_0$  and therefore by the Schur criterion [Te, Lemma 0.32] and the symmetry of the integral kernel the operator  $R_1$  is bounded. Let  $\mathbb{1}_\mu$  stand for the identity function from  $L_\mu^2$ . Note that the integral operator  $R_2$  is a rank-one operator  $C \mathbb{1}_\mu(\cdot, \mathbb{1}_\mu)_{L_\mu^2}$ . Consequently,  $R_2$  is also bounded. Now boundedness of  $R$  follows from decomposition  $R = R_1 + R_2$  and boundedness of  $R_1$  and  $R_2$  separately.  $\square$

After these preliminaries we are ready to analyze the operator-valued function  $\mathbb{R}_+ \ni k \mapsto Q(-k^2)$ . First, let us note that for a given  $k$  the operator  $Q(-k^2)$  is bounded in  $L_\mu^2$ . The proof of this fact can be done via repeating the argument from [BEKS94, Corollary 2.2]. Now our aim is to expand  $Q(\cdot)$  in a neighbourhood of zero.

**Proposition 2.8.** *Let  $\mu$  be a compactly supported positive Radon measure on  $\mathbb{R}^2$  from the generalized Kato class, and the operator-valued function  $Q(\cdot)$*

be defined as in (2.5). Then  $Q(\cdot)$  admits the expansion

$$(2.15) \quad Q(-k^2) = -\ln(k)P + R + O(k^2 \ln(k)), \quad k \rightarrow 0+,$$

in the operator norm, where  $P$  is a rank-one operator given by

$$(2.16) \quad P := \frac{1}{2\pi} \mathbb{1}_\mu(\cdot, \mathbb{1}_\mu)_{L_\mu^2}$$

and  $R$  is a bounded operator in  $L_\mu^2$  defined by

$$(2.17) \quad Rf := \frac{1}{2\pi} \int_{\mathbb{R}^2} \left( -\ln|\cdot - y| + C_E + \ln 2 \right) f(y) d\mu(y);$$

$C_E$  stands for the Euler-Mascheroni constant<sup>1</sup>, i.e.  $C_E = 0.57721\dots$

*Proof.* To prove the statement we employ the following expansion of the Macdonald function

$$(2.18) \quad K_0(x) = -\ln(x/2) + C_E + s(x), \quad x \rightarrow 0+,$$

where  $s(x) = O(x^2 \ln(x))$ , see [AS64, Equation 9.6.13]. In view of (2.6) and the compactness of the support of  $\mu$  the operator  $Q(-k^2)$  can be expanded into the sum of the rank-one operator  $-\ln(k)P$ , the operator  $R$  and the remaining operator  $S(k)$  with the integral kernel  $s(k|x - y|)$ . Since  $Q(-k^2)$ ,  $P$  and  $R$  are bounded the operator  $S(k)$  is bounded as well. Further, note that for sufficiently small  $k > 0$

$$|s(k|x - y|)| \leq A_\mu k^2 |\ln(k)|, \quad x, y \in \text{supp } \mu,$$

with some constant  $A_\mu > 0$ , which depends on  $\mu$ . Thus by Schur criterion the operator  $S(k)$  in  $L_\mu^2$  with the integral kernel  $s(k|x - y|)$  satisfies

$$\|S(k)\| = O(k^2 \ln k), \quad k \rightarrow 0+,$$

which completes the proof.  $\square$

*Remark 2.9.* Similar decomposition of the function  $Q(\cdot)$  is employed in [CK11] for some other purposes in the case of Dirac measure supported by a non-compact curve.

In the next lemma we gather some useful properties of the operator-valued function  $Q(\cdot)$ .

**Lemma 2.10.** *Let the operator-valued function  $Q(\cdot)$  be defined as in (2.5). Then the following statements hold.*

- (i)  $Q(-k^2) \geq 0$  for all  $k > 0$ .
- (ii)  $Q(-k_1^2) \leq Q(-k_2^2)$  for  $k_1 \geq k_2$ .

---

<sup>1</sup>This constant can be computed as  $C_E = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln(n) \right)$ .

- (iii) For any  $\varepsilon > 0$  there exists sufficiently small  $k > 0$  such that the spectrum  $\sigma(Q(-k^2))$  decomposes into two disjoint parts

$$\sigma_0(Q(-k^2)) \subset (0, \|R\| + \varepsilon)$$

with  $R$  as in (2.17) and

$$\sigma_1(Q(-k^2)) = \{\gamma(k)\},$$

where  $\gamma(k)$  is the eigenvalue of  $Q(-k^2)$  with multiplicity one.

- (iv) The function  $\gamma(\cdot)$  is continuous, strictly decaying, and  $\gamma(k) \rightarrow +\infty$  as  $k \rightarrow 0+$ .

*Proof.* The item (i) follows directly from the non-negativity of the Macdonald function and the representation of the integral kernel of  $Q(-k^2)$  given by (2.6).

The Macdonald function is monotonously decaying function of its argument, which yields the statement of (ii).

Note that according to Proposition 2.8 the function  $k \mapsto T(k)$ ,  $k > 0$  defined by

$$T(-k^2) := -\frac{2\pi}{\mu_T \ln k} Q(-k^2)$$

determines a realization of the operator family considered in Theorem 2.6 with  $\mathcal{H} = L_\mu^2$ ,  $\varphi = \frac{\mathbb{1}_\mu}{\sqrt{\mu_T}}$  and  $T_1 = -\frac{2\pi}{\mu_T} R$  with  $R$  as in (2.17). Thus for sufficiently small  $k > 0$  the spectrum of the operator  $Q(-k^2)$  can be separated into two parts as claimed in (iii) and the function  $\gamma(\cdot)$  is continuous. In view of (ii) the function  $\gamma(\cdot)$  is non-increasing. Suppose that for some  $k_1 < k_2$  the condition  $\gamma(k_1) = \gamma(k_2)$  holds, that implies  $\gamma(k) = c > 0$  for  $k \in [k_1, k_2]$ . Hence, by Proposition 2.3 we have  $[-k_1^2, -k_2^2] \subset \sigma_p(H_{(1/c)\mu})$ , which is a contradiction, because the point spectrum of any self-adjoint operator should be a countable set. This proves strict decay of  $\gamma(\cdot)$ .  $\square$

**2.4. Properties of the  $R_{\mu dx}(\cdot)$ -function.** In this subsection we investigate some properties of the operator-valued function  $R_{\mu dx}(\cdot)$  defined by (2.4). The unitary Fourier transform  $\mathcal{F} : L^2 \rightarrow L^2$  is defined as the extension by continuity of the integral transform

$$(\mathcal{F}f)(p) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ipx} f(x) dx, \quad f \in L^2 \cap L^1.$$

It is well-known that  $\mathcal{F}$  can be further extended by continuity up to the space  $\mathcal{S}'$ , cf. [AH91, Chapter 1.1.7]. Without a danger of confusion we keep the same notation  $\mathcal{F} : \mathcal{S}' \rightarrow \mathcal{S}'$  for this extension. In the following we will use also the abbreviation  $\mathcal{F}f = \widehat{f}$ ,  $f \in \mathcal{S}'$ . Applying again the standard results concerning the Sobolev spaces, see [AH91, Chapter 1.2.6], we can write

$$(2.19) \quad H^k = \{f \in \mathcal{S}' : \widehat{f}(p)(p^2 + 1)^{k/2} \in L^2\},$$

where the norm  $\|\cdot\|_k$  in  $H^k$  is defined by  $\|f\|_k = \|\widehat{f}(p)(p^2 + 1)^{k/2}\|$ . We define the functional  $\varphi\mu$  for  $\varphi \in L_\mu^2$  as

$$(\varphi\mu)(f) := \int_{\mathbb{R}^2} (J_\mu f)(x) \overline{\varphi(x)} d\mu(x), \quad f \in H^1,$$

with  $J_\mu$  as in Subsection 2.1. Let us show that  $\varphi\mu \in H^{-1}$ . Indeed for any  $f \in H^1$  we get

$$|(\varphi\mu)(f)| \leq \int_{\mathbb{R}^2} |(J_\mu f)(x)| |\varphi(x)| d\mu(x) \leq \|J_\mu f\|_{L_\mu^2} \|\varphi\|_{L_\mu^2} \leq C \|f\|_1 \|\varphi\|_{L_\mu^2}$$

with some constant  $C > 0$ , where we applied Hölder inequality in between and used that the embedding  $J_\mu$  of  $H^1$  into  $L_\mu^2$  is continuous. We have shown that the functional  $\varphi\mu$  is continuous on  $H^1$  and hence  $\varphi\mu \in H^{-1}$ . Further, we define

$$(2.20) \quad \widehat{\varphi}(p) := (\mathcal{F}(\varphi\mu))(p) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ipx} \varphi(x) d\mu(x), \quad \varphi \in L_\mu^2.$$

In the next lemma we explore basic properties of the above transform.

**Lemma 2.11.** *Let  $\mu$  be a compactly supported positive finite Radon measure on  $\mathbb{R}^2$  from the generalized Kato class. Then for any  $\varphi \in L_\mu^2$  its Fourier transform  $\widehat{\varphi}$  given by (2.20) is a bounded and Lipschitz continuous function.*

*Proof.* Let  $\varphi \in L_\mu^2$ . Since the measure  $\mu$  is finite the inclusion  $L_\mu^2 \subset L_\mu^1$  holds. The boundedness of  $\widehat{\varphi}$  follows from the estimate

$$\|\widehat{\varphi}\|_{L^\infty} \leq \frac{1}{2\pi} \|\varphi\|_{L_\mu^1} < \infty.$$

It remains to show that  $\widehat{\varphi}$  is Lipschitz continuous. Let us choose arbitrary  $p_1, p_2 \in \mathbb{R}^2$ . Applying (2.20) we obtain

$$(2.21) \quad |\widehat{\varphi}(p_1) - \widehat{\varphi}(p_2)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} |e^{-ip_1x} - e^{-ip_2x}| \cdot |\varphi(x)| d\mu(x).$$

Using the fact that the function  $\mathbb{R} \ni t \mapsto e^{it}$  is Lipschitz continuous we estimate

$$(2.22) \quad |e^{-ip_1x} - e^{-ip_2x}| = |1 - e^{-i(p_2-p_1)x}| \leq L|x||p_2 - p_1|$$

with some constant  $L > 0$ . Plugging (2.22) into (2.21) and using compactness of  $\mu$  we get

$$|\widehat{\varphi}(p_1) - \widehat{\varphi}(p_2)| \leq L'|p_2 - p_1|$$

with some constant  $L' > 0$ . □

*Remark 2.12.* Using the representation (2.19) of the Sobolev spaces we can extend operator  $R(-k^2)$  to a larger space. To derive this extension we apply

$$(2.23) \quad R(-k^2) = \mathcal{F}^{-1} \frac{1}{|p|^2 + k^2} \mathcal{F} : L^2 \rightarrow L^2,$$

cf. [AH91]. Operator  $\frac{1}{|p|^2 + k^2} \mathcal{F}$  is bounded as the map acting from  $H^{-1}$  to  $L^2$  and, consequently, it can be extended by continuity to the whole space

$H^{-1}$ . This means that  $R(-k^2)$  admits the analogous extension. Note that  $R_{\mu dx}(-k^2)\varphi$  with  $\varphi \in L_\mu^2$  can be identified with the extension of  $R(-k^2)$  defined above applied to  $\varphi\mu \in H^{-1}$ .

In the next lemma we provide the Fourier representation of  $R_{\mu dx}(-k^2)$ .

**Lemma 2.13.** *Let  $\mu$  be a compactly supported positive finite Radon measure on  $\mathbb{R}^2$  from the generalized Kato class. The operator  $R_{\mu dx}(-k^2): L_\mu^2 \rightarrow L^2$  defined by (2.4) admits the representation*

$$(2.24) \quad R_{\mu dx}(-k^2)\varphi = \mathcal{F}^{-1} \frac{\widehat{\varphi}(p)}{|p|^2 + k^2}, \quad \varphi \in L_\mu^2,$$

where  $\widehat{\varphi}$  is given by (2.20) and  $\mathcal{F}^{-1}$  is the inverse Fourier transform on  $\mathbb{R}^2$ .

*Proof.* Combining the statements of Remark 2.12 and (2.20) we get the claim.  $\square$

Having in mind later purpose we investigate in the next proposition the properties of  $R_{\mu dx}(-k^2)$  as  $k \rightarrow 0+$ .

**Proposition 2.14.** *Let  $\mu$  be a compactly supported positive finite Radon measure on  $\mathbb{R}^2$  from the generalized Kato class. Let the operator-valued function  $R_{\mu dx}(-k^2): L_\mu^2 \rightarrow L^2$  be as in (2.4). Then for any  $\varphi \in L_\mu^2$  the following asymptotics holds*

$$k^2 \|R_{\mu dx}(-k^2)\varphi\|_{L^2}^2 = \pi |\widehat{\varphi}(0)|^2 + O(\sqrt{k}), \quad k \rightarrow 0+,$$

where  $\widehat{\varphi}$  is the transform of  $\varphi$  defined by (2.20).

*Proof.* Let  $\varphi \in L_\mu^2$  and  $k > 0$ . Using Lemma 2.13 and applying the fact that  $\mathcal{F}^{-1}$  is unitary in  $L^2$  we obtain

$$(2.25) \quad k^2 \|R_{\mu dx}(-k^2)\varphi\|^2 = k^2 \int_{\mathbb{R}^2} \frac{|\widehat{\varphi}(p)|^2}{(|p|^2 + k^2)^2} dp = \int_{\mathbb{R}^2} \frac{|\widehat{\varphi}(kt)|^2}{(|t|^2 + 1)^2} dt.$$

For given  $\varepsilon > 0$  we disjoin the last integral in (2.25) onto regions

$$\mathcal{B}_k = \{t \in \mathbb{R}^2: |t| < \frac{1}{\sqrt{k}}\} \quad \text{and} \quad \mathcal{B}_k^c = \mathbb{R}^2 \setminus \overline{\mathcal{B}_k}.$$

Using boundedness of  $\widehat{\varphi}$  we obtain that

$$(2.26) \quad \begin{aligned} \int_{\mathcal{B}_k^c} \frac{|\widehat{\varphi}(kt)|^2}{(|t|^2 + 1)^2} dt &\leq C \int_{\mathcal{B}_k^c} \frac{1}{(|t|^2 + 1)^2} dt \\ &= C' \int_{\frac{1}{\sqrt{k}}}^{+\infty} \frac{r}{(r^2 + 1)^2} dr = O(k), \quad k \rightarrow 0+. \end{aligned}$$

Using boundedness and continuity of  $\widehat{\varphi}$ , and applying mean-value theorem we arrive at

$$(2.27) \quad \int_{\mathcal{B}_k} \frac{|\widehat{\varphi}(kt)|^2}{(|t|^2 + 1)^2} dt = |\widehat{\varphi}(\theta)|^2 \int_{\mathcal{B}_k} \frac{dt}{(|t|^2 + 1)^2},$$

where  $\theta \in \mathbb{R}^2$  and  $|\theta| \leq \sqrt{k}$ . Applying the asymptotic behaviour

$$\int_{\mathcal{B}_k} \frac{dt}{(|t|^2 + 1)^2} = \int_{\mathbb{R}^2} \frac{dt}{(|t|^2 + 1)^2} + O(k) = \pi + O(k), \quad k \rightarrow 0+,$$

to the formula (2.27) we obtain

$$\int_{\mathcal{B}_k} \frac{|\widehat{\varphi}(kt)|^2}{(|t|^2 + 1)^2} dt = \pi |\widehat{\varphi}(\theta)|^2 + O(k), \quad k \rightarrow 0+.$$

Lipschitz continuity of  $\widehat{\varphi}$  combined with the above formula, (2.25), (2.26) and  $|\theta| \leq \sqrt{k}$  imply that

$$k^2 \|R_{\mu dx}(-k^2)\varphi\|_{L^2}^2 = \pi |\widehat{\varphi}(0)|^2 + O(\sqrt{k}), \quad k \rightarrow 0+,$$

and the claim is proven.  $\square$

### 3. WEAKLY COUPLED BOUND STATE

In Subsection 3.1 we show that for sufficiently small coupling constant  $\alpha > 0$  the discrete spectrum of the self-adjoint operator  $H_{\alpha\mu}$  consists of exactly one negative eigenvalue of multiplicity one and we compute the asymptotics of this eigenvalue as  $\alpha \rightarrow 0+$ . Moreover, in Subsection 3.2 we compute the asymptotics of the corresponding eigenfunction in the same limit.

**3.1. Asymptotics of weakly coupled bound state.** In this subsection we compute the asymptotics of weakly coupled bound state. The technique we employ here is slightly different than the one applied in [Si76]. As a benefit it allows to include also regular potentials with stronger singularities.

**Theorem 3.1.** *Let  $\mu$  be a compactly supported positive finite Radon measure on  $\mathbb{R}^2$  from the generalized Kato class. Let the self-adjoint operator  $H_{\alpha\mu}$  be as in Definition 2.2. Then for all sufficiently small  $\alpha > 0$  the condition*

$$\sharp\sigma_d(H_{\alpha\mu}) = 1$$

*holds and the corresponding unique eigenvalue  $\lambda(\alpha) < 0$  satisfies*

$$(3.1) \quad \lambda(\alpha) \rightarrow 0- \quad \text{for } \alpha \rightarrow 0+.$$

*Proof.* We rely on the Birman-Schwinger principle from Proposition 2.3. In order to recover the eigenvalues of  $H_{\alpha\mu}$  we will investigate the following condition  $1 \in \sigma_p(\alpha Q(-k^2))$ . Let  $\sigma_i(Q(-k^2))$ ,  $i = 0, 1$  be as in Lemma 2.10 (iii). The possibility  $1/\alpha \in \sigma_0(Q(-k^2))$  for  $k > 0$  small enough is excluded due to Lemma 2.10 (iii). On the other hand,  $1/\alpha \in \sigma_1(Q(-k^2))$  is equivalent to the equation

$$\gamma(k) = 1/\alpha,$$

which in view of Lemma 2.10 (iv) has exactly one solution  $k(\alpha)$  for  $\alpha > 0$  small enough and moreover  $k(\cdot)$  satisfies

$$k(\alpha) \rightarrow 0+, \quad \alpha \rightarrow 0+.$$

Consequently,  $\lambda(\alpha) = -k(\alpha)^2$  gives the unique negative simple eigenvalue of  $H_{\alpha\mu}$  and the limiting property (3.1) holds.  $\square$

Our next aim is to derive asymptotics of  $\lambda(\alpha)$  for  $\alpha \rightarrow 0+$ .

**Theorem 3.2.** *Let  $\mu$  be a compactly supported positive finite Radon measure on  $\mathbb{R}^2$  from the generalized Kato class, and let  $H_{\alpha\mu}$  be the self-adjoint operator as in Definition 2.2. Then the eigenvalue of  $H_{\alpha\mu}$  admits the following asymptotics*

$$(3.2) \quad \lambda(\alpha) = -(C_\mu + o(1)) \exp\left(-\frac{4\pi}{\alpha\mu_T}\right), \quad \alpha \rightarrow 0+,$$

with

$$(3.3) \quad C_\mu = \exp\left(\frac{4\pi}{\mu_T^2}(R\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2}\right),$$

where  $R$  is defined in (2.17).

*Proof.* Let us consider the operator-valued function

$$(3.4) \quad T(k) := -\frac{2\pi}{\mu_T \ln k} Q(-k^2), \quad k > 0,$$

where  $Q(\cdot)$  is defined by (2.5). Comparing the expansion from Proposition 2.8 and the definition (3.4) one can see that the operator-valued function  $T(\cdot)$  reflects the structure assumed in Theorem 2.6; precisely  $\mathcal{H} = L_\mu^2$  and

$$T_0 = \varphi_\mu(\cdot, \varphi_\mu)_{L_\mu^2}, \quad T_1 = -\frac{2\pi}{\mu_T} R,$$

where  $\varphi \equiv \varphi_\mu := \frac{\mathbb{1}_\mu}{\sqrt{\mu_T}}$ . Therefore, for sufficiently small  $k > 0$  the spectrum of  $T(k)$  can be separated into two disjoint parts:  $\sigma_0(k)$  located in the neighborhood of 0 and  $\sigma_1(k)$  consisting of exactly one simple eigenvalue  $\omega(k)$  located in the neighborhood of 1 and admitting the asymptotic expansion

$$\omega(k) = 1 + \frac{1}{\ln k}(T_1\varphi_\mu, \varphi_\mu)_{L_\mu^2} + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

Applying the definition of  $T_1$  to the last expansion, we arrive at

$$(3.5) \quad \omega(k) = 1 - \frac{2\pi}{\mu_T^2 \ln k}(R\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2} + O\left(\frac{1}{\ln^2 k}\right), \quad k \rightarrow 0+.$$

Suppose that  $\alpha > 0$  is sufficiently small, so that  $\sharp\sigma_d(H_{\alpha\mu}) = 1$ , cf. Theorem 3.1. Let  $\lambda(\alpha) = -k^2(\alpha)$  standardly denote the corresponding unique eigenvalue of  $H_{\alpha\mu}$  which in view of Theorem 3.1 converges as  $\lambda(\alpha) \rightarrow 0-$  for  $\alpha \rightarrow 0+$ . Combining the Birman-Schwinger principle together with the definition of  $T(\cdot)$  we obtain the following condition

$$-\frac{1}{2\pi}\alpha\mu_T\omega(k(\alpha))\ln k(\alpha) = 1$$

for the value  $k(\alpha)$ . Applying to the above equation the asymptotic expansion of  $\omega(\cdot)$  given by (3.5) we get

$$-\frac{\alpha\mu_T \ln k(\alpha)}{2\pi} + \frac{\alpha}{\mu_T} (R\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2} + O\left(\frac{\alpha}{\ln k(\alpha)}\right) = 1, \quad \alpha \rightarrow 0+.$$

The latter is equivalent to

$$(3.6) \quad \ln k(\alpha) = -\frac{2\pi}{\alpha\mu_T} + \frac{2\pi}{\mu_T^2} (R\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2} + o(1), \quad \alpha \rightarrow 0+,$$

which yields

$$\lambda(\alpha) = -k(\alpha)^2 = -(C_\mu + o(1))e^{-\frac{4\pi}{\alpha\mu_T}}, \quad \alpha \rightarrow 0+,$$

with  $C_\mu$  as in (3.3).  $\square$

**Example 3.1.** We will test the above theorem on a special model. Namely, let  $\mu$  be defined via a Dirac measure supported on a circle  $C_r$  of radius  $r$ ; precisely

$$(3.7) \quad \mu(\Omega) = l(\Omega \cap C_r),$$

where  $l(\cdot)$  is the one-dimensional measure defined by the length of the arc. This example was already studied in [ET04], where the authors compute negative spectrum of  $H_{\alpha\mu}$  (with  $\mu$  as above) using separation of variables.

In order to recover the asymptotic behavior of the eigenvalue of  $H_{\alpha\mu}$  with  $\alpha$  small and  $\mu$  defined by (3.7), we will compute the constant  $C_\mu$  given in (3.3). According to [KV12, Lemma 3.2] we obtain

$$(3.8) \quad (Q(-k^2)\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2} = 2\pi r^2 \int_0^\infty \frac{|J_0(y)|^2 y}{(kr)^2 + y^2} dy,$$

where  $J_0(\cdot)$  is the Bessel function of order 0. Applying [GR, Equation 6.535] in the above formula we arrive at

$$(3.9) \quad (Q(-k^2)\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2} = 2\pi r^2 I_0(kr) K_0(kr).$$

Using the asymptotic expansions [AS64, 9.6.12, 9.6.13]

$$I_0(x) = 1 + O(x^2), \quad x \rightarrow 0+,$$

$$K_0(x) = (-\ln(x/2) + C_E) + O(x^2 \ln x), \quad x \rightarrow 0+,$$

of  $I_0(\cdot)$  and  $K_0(\cdot)$  in the neighbourhood of zero, we obtain

$$(3.10) \quad I_0(kr) K_0(kr) = -\ln \frac{kr}{2} + C_E + O(k), \quad k \rightarrow 0+.$$

Combining equations (3.9) and (3.10) we get

$$(3.11) \quad (Q(-k^2)\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2} = 2\pi r^2 \left( -\ln \frac{kr}{2} + C_E + O(k) \right), \quad k \rightarrow 0+.$$

The decomposition stated in Proposition 2.8 yields

$$(3.12) \quad (R\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2} = (Q(-k^2)\mathbb{1}_\mu, \mathbb{1}_\mu)_{L_\mu^2} + 2\pi r^2 \ln k + O(k^2 \ln k), \quad k \rightarrow 0+.$$



In fact, the left hand side of (3.12) does not depend on  $k$ . Consequently, inserting (3.11) into (3.12) and taking the limit  $k \rightarrow 0+$  we get

$$(R\mathbb{1}_\mu, \mathbb{1}_\mu)_{L^2_\mu} = 2\pi r^2 \left( -\ln \frac{r}{2} + C_E \right).$$

In view of (3.3) this implies that  $C_\mu = \frac{4}{r^2} \exp(2C_E)$  and finally

$$\lambda(\alpha) = -\frac{4}{r^2} e^{2C_E} e^{-\frac{2}{\alpha r}} (1 + o(1)), \quad \alpha \rightarrow 0+,$$

which is fully consistent with a result of [ET04, Subsection 2.1] and furthermore refines that result.

*Remark 3.3.* Following the line of [BEKS94] one can introduce a sign changing weight in  $\gamma \in L^\infty(\mathbb{R}^2)$  and consider more general operators defined via quadratic forms

$$\mathfrak{q}_{\alpha\gamma\mu}[f] := \|\nabla f\|_{L^2}^2 - \alpha \int_{\mathbb{R}^2} \gamma(x) |f(x)|^2 d\mu(x), \quad \text{dom } \mathfrak{q}_{\alpha\gamma\mu} = H^1.$$

In this case one can get the asymptotics similar to (3.2) with  $\gamma$  involved. Instead of  $\mu(\mathbb{R}^2)$  in the exponent there will be  $I := \int_{\mathbb{R}^2} \gamma(x) d\mu(x) > 0$ . The asymptotics could be different if  $I = 0$ . This case requires special analysis.

**3.2. Asymptotics of the eigenfunction corresponding to the weakly coupled bound state.** As we have shown in the previous section the operator  $H_{\alpha\mu}$  has exactly one negative eigenvalue for sufficiently small  $\alpha > 0$ . The aim of this section is to recover the asymptotic behaviour of the corresponding eigenfunction in the limit  $\alpha \rightarrow 0+$ .

**Theorem 3.4.** *Let  $\mu$  be a compactly supported positive finite Radon measure on  $\mathbb{R}^2$  from the generalized Kato class. Let  $\lambda(\alpha)$  be the unique eigenvalue of  $H_{\alpha\mu}$  for  $\alpha \rightarrow 0+$  and  $-k_\alpha^2 = \lambda(\alpha)$ . Then the corresponding eigenfunction has the form*

$$f_\alpha(\cdot) = \frac{k_\alpha}{2\pi} \int_{\mathbb{R}^2} K_0(k_\alpha |\cdot - y|) d\mu(y) + O\left(\frac{1}{\ln k_\alpha}\right), \quad \alpha \rightarrow 0+,$$

where the error term is understood in the sense of  $L^2$ -norm; moreover the  $L^2$ -norm of  $f_\alpha$  has non-zero finite limit as  $\alpha \rightarrow 0+$ .

*Proof.* In the proof of this theorem we rely on Proposition 2.3. For non-trivial  $\phi_\alpha \in \ker(I - \alpha Q(-k_\alpha^2))$  the function

$$(3.13) \quad g_\alpha(\cdot) := R_{\mu dx}(-k_\alpha^2)\phi_\alpha = \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(k_\alpha |\cdot - y|) \phi_\alpha(y) d\mu(y),$$

reproduces the eigenfunction of  $H_{\alpha\mu}$ . Similarly as in the proof of Theorem 3.2 we conclude that  $\phi_\alpha$  is an eigenfunction of the operator

$$T(k_\alpha) = -\frac{2\pi}{\mu_T \ln(k_\alpha)} Q(-k_\alpha^2)$$

corresponding to the eigenvalue  $-\frac{2\pi}{\mu_T \ln(k_\alpha)\alpha}$ . Recall that the family  $\mathbb{R}_+ \ni k \mapsto T(k)$  is a realization of the operator family considered in Theorem 2.6 with  $\mathcal{H} = L_\mu^2$ ,  $\varphi := \frac{\mathbb{1}_\mu}{\sqrt{\mu_T}}$ ,  $T_0 := \varphi(\cdot, \varphi)$ ,  $T_1 := -\frac{2\pi}{\mu_T}R$  and  $R$  as in (2.17). Hence, by Theorem 2.6 (iii) we obtain that  $\phi_\alpha$  can be chosen in the form

$$(3.14) \quad \phi_\alpha = \mathbb{1}_\mu + \varsigma_\alpha, \quad \text{where} \quad \|\varsigma_\alpha\|_{L_\mu^2} = O\left(\frac{1}{\ln k_\alpha}\right) \quad \text{as} \quad \alpha \rightarrow 0+.$$

By Proposition 2.14 we obtain

$$(3.15) \quad k_\alpha^2 \|R_{\mu dx}(-k_\alpha^2)\mathbb{1}_\mu\|^2 = \frac{\mu_T^2}{4\pi} + O(\sqrt{k_\alpha}), \quad \alpha \rightarrow 0+,$$

where we used that  $\widehat{\mathbb{1}}_\mu(0) = \frac{1}{2\pi}\mu_T$ . Hölder inequality yields

$$(3.16) \quad |\widehat{\varsigma}_\alpha(0)| \leq \frac{1}{2\pi} \|\varsigma_\alpha\|_{L_\mu^1} \leq \frac{1}{2\pi} \|\varsigma_\alpha\|_{L_\mu^2} \sqrt{\mu_T}.$$

Hence, using Proposition 2.14, (3.14) and (3.16) we get

$$(3.17) \quad k_\alpha^2 \|R_{\mu dx}(-k_\alpha^2)\varsigma_\alpha\|^2 = O\left(\frac{1}{\ln^2 k_\alpha}\right) \quad \alpha \rightarrow 0+,$$

According to (3.13), (3.14), (3.15) and (3.17)

$$f_\alpha := k_\alpha R_{\mu dx} \phi_\alpha$$

is an eigenfunction of  $H_{\alpha\mu}$  and satisfies

$$\|f_\alpha\|_{L^2} \rightarrow \frac{\mu_T}{2\sqrt{\pi}}, \quad \text{as} \quad \alpha \rightarrow 0+,$$

moreover

$$f_\alpha(\cdot) = \frac{k_\alpha}{2\pi} \int_{\mathbb{R}^2} K_0(k_\alpha|\cdot - y|) d\mu(y) + O\left(\frac{1}{\ln k_\alpha}\right), \quad \alpha \rightarrow 0+,$$

holds, and the claim is proven.  $\square$

**3.3. Concluding remarks.** The asymptotics of the unique eigenvalue as well as the corresponding eigenfunction were proved for the compactly supported measure  $\mu$ . The assumption of the compactness was essential, for example, for the decomposition (2.15) which was the fundamental tool for the proof of Theorem 3.2. It seems that the most natural way is to apply approaching of non-compactly supported measure  $\mu$  by an appropriate sequence  $\mu_n$  of compactly supported measures. However, we face the problem that the error term  $o(1)$  appearing in (3.2) is not, generally, uniform with respect to  $n$ .

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